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AEOLIAN TONES OF A HONEYCOMB LATTICE CELL

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Aeroacoustic resonant oscillations (aeolian tones) are studied for flow past two plates forming a cross in a square cross section channel. Possible oscillation modes are classified on the basis of admissible symmetry groups and the existence of the modes is proved. The infinite linear system of equations for these modes obtained by the sewing method was simplified and studied numerically. Curves of eigenfrequency versus plate length are constructed. The form of the eigenfunctions is studied.

1. Formulation of the Problem. A square cross section channel is considered. After the variables are made nondimensional by the formulas [1]

$$x = \frac{x'}{\sqrt{1-M^2}\,H}, \quad y = \frac{y'}{H}, \quad z = \frac{z'}{H}, \quad \lambda = \frac{\omega H}{c\sqrt{1-M^2}}, \quad M = \frac{U}{c} \tag{1}$$

(H is the height of the channel, c is the speed of sound in the medium, ω is the circular frequency of the oscillations, and U is the flow velocity; prime denotes dimensional variables), the height and width of the channel become equal to unity (Fig. 1). Two identical perpendicular plates of length b and unit width are placed in the channel. The line of intersection of the plates halves them. The coordinate origin is at the center of the plates, B is the boundary of the channel, Γ are the plate profiles, Ω is the region occupied by a gas, and $\Omega' = \Gamma \cup B$.

A homogeneous flow moves at velocity U in the channel. The flow past the plates can give rise to resonant aeroacoustic oscillations due to the formation and shedding of ordered vortical structures from the plate edges. The solution of the linearized equations of motion for the gas can be represented by the way sum of vortical and acoustic modes in the region occupied by the gas [2]. This representation does not hold only at the vortex shedding edge [3]. It can be assumed that the unknown singularity at the shedding edge is described by a vortical mode, the acoustic oscillations are due only to a vortical mode, and the effect of acoustic waves on the sound source should be taken into account only for the gas flow regimes leading to acoustic resonant phenomena [3]. In a coordinate system attached to the plates, the acoustic and vortical oscillations are steady in time with a certain ordered vortical structure [3].

Unperturbed acoustic oscillations in the coordinate system $OXYZ$ (Fig. 1) are described by the potential $u(x, y, z)$ of the acoustic perturbation of the main gas flow velocity, which should satisfy the following system of equations [1, 3]:

$$\Delta u + \lambda^2 u = 0 \qquad \text{in} \quad \Omega,$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad B \cup \Gamma, \qquad \int\limits_{\Omega_0} \Big(u^2 + (\nabla u)^2\Big)\, d\Omega < \infty \quad \forall\, \Omega_0 \subset \Omega. \tag{2}$$

Below, the problem (2) will be called the NO problem (natural oscillation problem). The formulation of the problem is discussed in [3, 4].

Honeycomb lattices are widely used in aerodynamics to straighten gas flows. The structure studied in the present paper simulates a unit cell of a honeycomb lattice for plates.

The main difficulty of the problem is that the discrete spectrum of the problem is embedded in the absolutely continuous spectrum of the operator $-\Delta$, which occupies the nonnegative part of the real axis.

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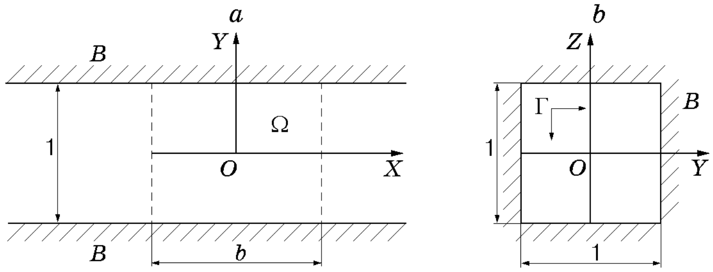

Fig. 1. Parameters of the channel and plates: (a) the section along the channel; (b) the cross section.

TABLE 1

One-Dimensional Representation of the Group D_4

Transformation	ψ_1	ψ_2	ψ_3	ψ_4
r^k	1	1	$(-1)^k$	$(-1)^k$
sr^k	1	-1	$(-1)^k$	$(-1)^{k+1}$

2. Classification of Possible Natural Oscillations. Because of the symmetry of the problem, the space of admissible solutions can be restricted. In this space, the origin of the absolutely continuous spectrum of the operator $-\Delta$ is located to the right of the zero point.

If the natural oscillations are even with respect to the variables y and z, then the solution and its derivative do not suffer discontinuity on the planes OXZ and OXY, in which the plates are located. Therefore, the plates do not influence the oscillations, i.e., such oscillations should exist in the channel in the absence of plates. However, this is impossible because in this case natural oscillations that are even with respect to the variables y and z do not occur.

Oscillations that are even with respect to one plate and odd with respect to the other plate are described by the problem with one plate in a channel considered in [4].

The cross section of the structure admits symmetry (Fig. 1b) described only by the dihedral group D_4 and the group C_4 [5] (the remaining groups describing the symmetry admitted by the structure are only subgroups of the groups D_4 and C_4).

The generating elements of the group D_4 are rotation r through $\pi/4$ about the OX axis and mirror symmetry s about the plane OXZ. The characters ψ_1, ψ_2, ψ_3, and ψ_4 of the one-dimensional (irreducible) representations of the group D_4 are present in Table 1 [5]. Each character defines the structure of one solution. Oscillations with characters ψ_1 and ψ_3 correspond to oscillations that are even with respect to the variables y and z. As shown above, such oscillations cannot exist.

The group C_4, which describes rotational symmetry about the OX axis, defines the representation of the eigenfunctions in the form of Rayleigh–Bloch waves for $j = 0$, 1, 2, and 3: $C_4\langle u(x, y, z)\rangle = \exp(i\pi j/2)u(x, y, z)$. For $j \neq 0$, Rayleigh–Bloch waves describe traveling waves in the plane OYZ with a phase shift of $\exp(i\pi j/2)$ at the points (x, y, z) and $(x, -z, y)$. The cases $j = 0$ and 2 are already considered in the one-dimensional representations of the group D_4. From a physical viewpoint, the traveling modes with $j = 1$ and $j = 3$ differ only in propagation direction (counter-clockwise and clockwise, respectively, in the plane OYZ), i.e., these oscillations should correspond to identical frequencies.

The problem possesses mirror symmetry about the plane OYZ. Therefore, the space of admissible solutions is represented as the direct sum of solutions that are even and odd with respect to the variable x. Below, even and odd oscillations imply even and odd oscillations with respect to the variable x.

From the aforesaid it follows that there are three oscillation mode that differ from oscillations near one plate in the channel [ignoring evenness (oddness) with respect to x]: 1) mode with the character ψ_2 (α-mode); 2) Rayleigh–Bloch wave with $j = 1$ (β-mode); 3) mode with the character ψ_4 (γ-mode).

We note that for α-modes, an absolutely continuous spectrum begins at the point $10\pi^2$, for β-modes, it begins at the point π^2, and for γ-modes, at the point $2\pi^2$. Below, σ_0^2 denotes the beginning of a continuous spectrum.

3. Form of Natural Oscillations with Allowance for Symmetry. For $x \geqslant b/2$, the region of the channel is denoted by Ω_1, and for $x \leqslant -b/2$, it is denoted by Ω_6. For $-b/2 \leqslant x \leqslant b/2$, the first, second, third, and fourth quadrants in the plane OYZ are denoted by Ω_2, Ω_3, Ω_4, and Ω_5, respectively. In view of the symmetry conditions of the problem, it suffices to specify the form of the solution only in the regions Ω_1 and Ω_2. We designate these solutions by $u_1(x, y, z)$ and $u_2(x, y, z)$, respectively.

With allowance for symmetry, the α- and γ-modes have the form

$$
\begin{aligned}
u_1 &= \sum_{n,m}^{+\infty} b_{n,m} \Big[\sin(\pi(2n+1)y)\sin(\pi(2m+1)z) \\
&\qquad + (-1)^p \sin(\pi(2m+1)y)\sin(\pi(2n+1)z) \Big] \, \mathrm{e}^{-x\gamma(2m+1,2n+1)}, \\
u_2 &= \sum_{n,m}^{+\infty} a_{n,m} \Big[\cos(2\pi ny)\cos(2\pi mz) + (-1)^p \cos(2\pi my)\cos(2\pi nz) \Big] \left\{ \begin{array}{c} \cosh(-x\gamma_0) \\ \sinh(-x\gamma_0) \end{array} \right\},
\end{aligned} \tag{3}
$$

where $\gamma(m, n) = \sqrt{\pi^2 n^2 + \pi^2 m^2 - \lambda^2}$ and $\gamma_0 = \gamma(2n, 2m)$; for the α-modes, $p = 1$ and the summation begins with $n = 1$ and $m = 0$ for $n > m$; for the γ-modes, $p = 0$ and the summation begins with $n = 0$ and $m = 0$ for $n \geqslant m$. Here and below, the upper expression in braces corresponds to even modes and the lower expression to odd modes.

With allowance for symmetry, the β-modes have the following form ($n > m$ and $n + m$ is an odd number):

$$
\begin{aligned}
u_1 &= \sum_{n=1,m=0}^{+\infty} b_{n,m} \Big[\cos(\pi n(y - 1/2))\cos(\pi m(z - 1/2)) \\
&\qquad + (-1)^n i \cos(\pi m(y - 1/2))\cos(\pi n(z - 1/2)) \Big] \, \mathrm{e}^{-x\gamma(m,n)}, \\
u_2 &= \sum_{m,n=0}^{+\infty} a_{m,n} \cos(2\pi ny)\, \cos(2\pi mz) \left\{ \begin{array}{c} \cosh(-x\gamma_0) \\ \sinh(-x\gamma_0) \end{array} \right\}
\end{aligned} \tag{4}
$$

(i is an imaginary unity).

For the α-, β-, and γ-modes, the NO problem is called the NOS problem (natural oscillation problem with symmetry). We choose a cylindrical coordinate system $\{(\rho, \varphi, z)\colon \rho \geqslant 0,\ -\pi \leqslant \varphi \leqslant \pi,\ 0 \leqslant z \leqslant 1\}$ along the edge of one of the plates.

Lemma 1. *In the* NO *problem, the condition of finiteness of energy in the neighborhood of the plate edge is equivalent to the conditions* $u(\rho, \varphi, z) \simeq d(z) + \rho f(z)\cos\varphi$ *in the middle of the plate edge,* $u(\rho, \varphi, z) \simeq d(z) + \sqrt{\rho}\cos(\varphi/2) f(z)$ *for* $\rho \to 0$, *and* $d(z) \in W_2^1(\mathbb{R})$ *and* $f(z) \in W_2^1(\mathbb{R})$ *at the remaining points of the edge.*

Lemma 1 formulates conditions on the form of the solution of the NO problem that are equivalent to the condition at the edge [6].

Direct replacement of Cartesian coordinates in (3) and (4) by cylindrical coordinates proves the following lemma:

Lemma 2. The α- *and* γ*-modes have finite energy in the neighborhood of the edge. For the* β*-mode, the condition of finiteness of energy in the neighborhood of the edge is equivalent to the following relations* ($\forall\ k_1 \in \mathbb{Z}_+$):

$$
\sum_{k_1 \geqslant n} b_{2k_1+1,n}(-1)^n \, \mathrm{e}^{-b\gamma(2k_1+1,2n)/2} - i \sum_{n > k_1} b_{2n,2k_1+1}(-1)^n \, \mathrm{e}^{-b\gamma(2k_1+1,2n)/2} = 0. \tag{5}
$$

4. Existence of Natural Oscillations. We assume that apart from the boundary conditions of the NOS problem, for $R \geqslant b/2$, the Dirichlet condition $u(\pm R, y, z) = 0$ (DR) or the Neumann condition $u_x(\pm R, y, z) = 0$ (NR) is satisfied. Furthermore, for $R > b/2$, the solution is equal to zero in the region Ω. NOS problems with such conditions will be denoted by NOS (DR) and NOS (NR), respectively, and their eigenvalues by $\lambda_{\mathrm{D}R}$ and $\lambda_{\mathrm{N}R}$, respectively. From the "Dirichlet–Neumann bracket" principle, it follows that for all numbers R, the following inequalities are valid [7]:

$$
\lambda_{\mathrm{N}R}^k \leqslant \lambda_*^k \leqslant \lambda_{\mathrm{D}R}^k \tag{6}
$$

(the superscript k denotes the eigenvalue number).

Remark 1. If the strict inequalities $\lambda_{\mathrm{D}R} < \sigma_0$ and $0 < \lambda_{\mathrm{N}R}$ hold for some values $R \geqslant b/2$, the existence of eigenvalues of the NOS problem follows from (6) [7].

Remark 2. If $R > b/2$, then, by virtue of connectivity of the region and symmetry conditions, the solution cannot be a constant, and, hence, $0 < \lambda^1_{\mathrm{N}R}$.

Let $\Omega_R = \Omega \cap \{|x| < R\}$. As is known from the theory of variational methods [7],

$$(\lambda_{\mathrm{D}R})^2 \leqslant \int\limits_{\Omega_R} |\nabla u|^2 \, d\Omega_R \Bigg/ \int\limits_{\Omega_R} |u|^2 \, d\Omega_R = \mu^2(R) < \sigma_0^2. \tag{7}$$

Here $|u|$ is the modulus of a solution of a NOS problem if the solution is a complex-valued function.

The function $\mu^2(R)$ has the following asymptotic expansion in terms of R: $\sigma_0^2 + A/R + O(1/R^2)$. It is necessary to choose a function $u(x, y, z)$ satisfying the DR condition so that the inequality $\mu^2(R) < \sigma_0^2$ is satisfied, i.e., the variable A is negative for this function.

The solutions of NOS (DR) and NOS (NR) problems are rather smooth, suffer discontinuity only on the profiles, and can be written as $u = u_0 + ku_1$, where k is an arbitrary real number, u_0 is a generalized eigenfunction of the operator $-\Delta$ in the region Ω_R with the condition DR corresponding to the beginning of the continuous spectrum, and u_1 is a function that is discontinuous on Γ, continuous in the regions Ω_j $(j = 2, \ldots, 5)$, and equal to zero in the regions Ω_1 and Ω_6. From this representation for the function u it follows that $A = A(k) = A_1k^2 + A_2k$, and if $A_1 \neq 0$ and $A_2 \neq 0$, then k always exists such that $A(k) < 0$. Thus, the proof of the theorem of existence of oscillations reduces to search for functions u_1 with the required properties for each oscillation mode. The corresponding functions are given below.

α-Mode of Oscillations. The continuous component is

$$u_0 = (\sin(3\pi y)\sin(\pi z) - \sin(\pi y)\sin(3\pi z))\cos(\pi x/(2R)),$$

and the discontinuous component is $u_1 = k(y - z)\cos(\pi x/b)$ in Ω_2. From the asymptotic expansion (7) it follows that $A_1 = (48b^2 + \pi^2 - 10\pi^2 b^2)/(24b)$ and $A_2 = 1792b/(9\pi^2)$.

β-Mode of Oscillations. The continuous component is

$$u_0 = (\sin(\pi y) + i\sin(\pi z))\cos(\pi x/(2R)),$$

and the discontinuous component is $u_1 = k\cos(\pi x/b)$ in Ω_2. From the asymptotic expansion (7) it follows that $A_1 = -\pi^2(b^2 - 1)/(2b)$ and $A_2 = -8b$.

γ-Mode of Oscillations. The continuous component is

$$u_0 = \sin(\pi z)\sin(\pi y)\cos(\pi x/(2R)),$$

and the discontinuous component is $u_1 = k\cos(\pi x/b)$ in Ω_2. From the asymptotic expansion (7) it follows that $A_1 = 2\pi^2(1 - 2b^2)/b$ and $A_2 = -128b/\pi$.

Thus, we proved the following theorem:

Theorem 1. *In a square cross section channel, natural oscillations of the type of α-, β-, and γ-modes near a unit cell of a honeycomb lattice of plates exist always.*

Similarly [3], using inequality (6) for $R = b/2$ and the representations of solutions (3) and (4), we prove the following theorems:

Theorem 2. *The oscillation frequencies of the α-modes belong to the interval* $(2\pi, \sqrt{10}\pi)$.

Theorem 3. *The number K of the oscillation modes located below σ_0 satisfies the following inequalities* (*only integer solutions are used*):

— *for α-modes,* $\max(1, \sqrt{6}b - 1) \leqslant K < \sqrt{6}b + 1$;

— *for β-modes,* $\max(1, \sqrt{2}b - 1) \leqslant K < \sqrt{2}b + 1$;

— *for γ-modes,* $\max(1, b - 1) \leqslant K < b + 1$.

It should be noted that for all b, each inequality in the theorem has not more than two integer solutions.

5. Numerical Investigation of Natural Oscillations. In order that a function

$$u(x, y, z) = \begin{cases} u_1(x, y, z) & \text{in} \quad \Omega_1, \\ u_2(x, y, z) & \text{in} \quad \Omega_2 \end{cases}$$

be a solution of the NOS problem, the conditions of continuity of the solution and its normal derivative should be satisfied on the boundary between the regions Ω_1 and Ω_2 (sewing method) [8]

$$u_1 = u_2, \qquad \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} \quad \text{on} \quad \partial\Omega_1 \cap \partial\Omega_2. \tag{8}$$

Conditions (8) imply that a function $u(x, y, z)$ defined in the regions $\Omega_1, \ldots, \Omega_6$ is a weak solution, which, by virtue of the theory of elliptic equations, is a strong solution of the NOS problem.

Multiplying relation (8) by $\cos(2\pi n_1 y)\cos(2\pi m_1 z)$ and integrating the variables y and z from 0 to 1/2, we obtain two relations. Substituting one of these relations into the other and simplifying the result, we obtain the equation

$$\int_0^{1/2}\int_0^{1/2}\left(\left\{\begin{array}{l}\tanh(-x\gamma_1)\\ \coth(-x\gamma_1)\end{array}\right\} u_1 - \frac{1}{\gamma_1}\frac{\partial u_1}{\partial x}\right)\cos(2\pi m_1 z)\cos(2\pi n_1 y)\,dy\,dz = 0,$$

where $\gamma_1 = \gamma(2m_1, 2n_1)$ and $x = b/2$. The form of the function $u_1(x, y, z)$ and the choice of the variables m_1 and n_1 depend on which of the three cases is considered.

For the α and γ-modes, we obtain the system

$$\sum_{n,m}^{+\infty} b_{n,m}\,\mathrm{e}^{-b\gamma/2}\left((\gamma_1+\gamma)\,\mathrm{e}^{b\gamma_1/2} + (-1)^l(\gamma-\gamma_1)\,\mathrm{e}^{-b\gamma_1/2}\right)\Big(\frac{\pi(2n+1)}{(\pi(2n+1))^2-(2\pi n_1)^2}$$
$$\times\frac{\pi(2m+1)}{(\pi(2m+1))^2-(2\pi m_1)^2} + (-1)^p\frac{\pi(2n+1)}{(\pi(2m+1))^2-(2\pi n_1)^2}\,\frac{\pi(2m+1)}{(\pi(2n+1))^2-(2\pi m_1)^2}\Big) = 0, \tag{9}$$

where the summation begins with $n = 1$ and $m = 0$ for $n > m$ and $p = 1$ for α-modes and with $n = 0$ and $m = 0$ for $n \geqslant m$ and $p = 0$ for γ-modes. The variables n_1 and m_1 vary in the same ranges as the variables n and m; $\gamma = \gamma(2n + 1, 2m + 1)$.

For the β-modes, we have the following system ($n > m$ and $n + m$ is an odd number):

$$\sum_{n=1,m=0}^{+\infty} b_{n,m}\,\mathrm{e}^{-b\gamma(n,m)/2}\left((\gamma_1+\gamma)\,\mathrm{e}^{b\gamma_1/2} + (-1)^l(\gamma-\gamma_1)\,\mathrm{e}^{-b\gamma_1/2}\right)$$
$$\times\Big(g(n,n_1)g(m,m_1) + (-1)^m i g(m,n_1)g(n,m_1)). \tag{10}$$

Here n_1 and $m_1 \in \mathbb{Z}_+$ and $g(n,n_1) = \int_0^{1/2}\cos(\pi n(y-1/2))\cos(2\pi n_1 y)\,dy$.

In systems (9) and (10), the number l is even (odd) for even (odd) oscillations, respectively. The summation is taken over the subscripts m and n, i.e., the sums in the equations are double. These sums were reduced by two methods. In the first method, we used square partial sums, and in the second method, triangular partial sums based on Cantor numbering of pairs of eigenvalues [9]. The Cantor numbering of pairs of eigenvalues is a bijective function $C : \mathbb{Z}_+^2 \to \mathbb{Z}_+$. To solve the problem (2), it is necessary to use a modification of this mapping because in the modes considered, the set of subscripts of the coefficients is not the set $\mathbb{Z}_+^2$ but its subset.

According to Lemma 2, for the α- and γ-modes, no additional relations are required to satisfy the condition of finiteness of energy in the neighborhood of the edge, and for the β-modes, system (10) should be solved together with relations (5).

Numerical investigation of systems (9) and (10) shows that the first term in the sum corresponding to the least values of the variables n_1 and m_1 makes a major contribution to the magnitude of the eigenvalue. They are equal in both systems of equations. The first term is a simple approximation of the dispersion relation.

In the case of even α-, β-, and γ-modes, the first term is written as

$$\tan(\mu(\lambda)b/2) = \sqrt{\sigma_0^2 - \lambda^2}/\mu(\lambda), \tag{11}$$

and in the case of odd modes, it is written as

$$\tan(\mu(\lambda)b/2) = -\mu(\lambda)/\sqrt{\sigma_0^2 - \lambda^2}, \tag{12}$$

where $\mu(\lambda) = \sqrt{\lambda^2 - 4\pi^2}$ for the α-modes and $\mu(\lambda) = \lambda$ for the β- and γ-modes; σ_0 is the beginning of a continuous spectrum. These formulas lead to the following statements for the α-, β-, and γ-modes:

Statement 1. *For even (or odd) oscillations of one mode corresponding to frequency λ, the plate lengths b_1 and b_2 are linked by the relation $|b_1 - b_2| = 2\pi l/\mu(\lambda)$, where $l \in \mathbb{N}$.*

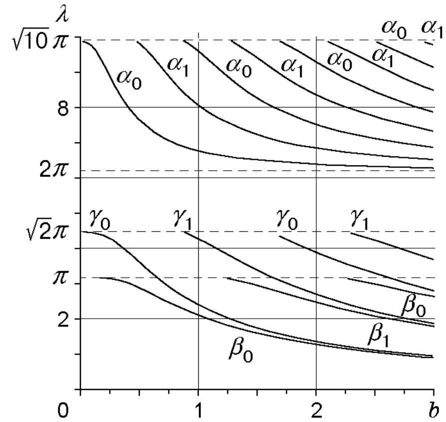

Fig. 2. Oscillation frequency versus plate length for the α-, β-, and γ-modes.

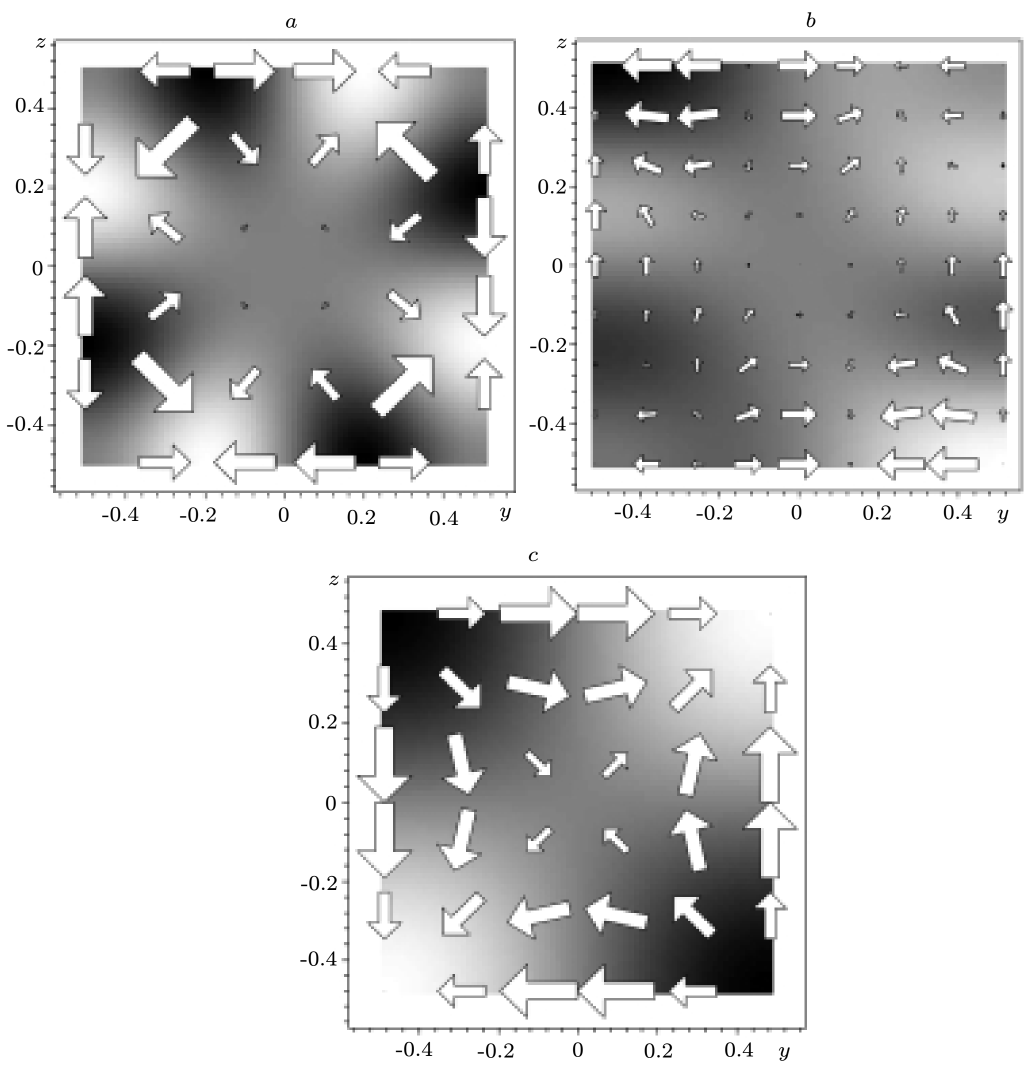

Fig. 3. Velocity and pressure fields for the α-mode (a), β-mode (b), and γ-mode (c).

Statement 2. *If the length b_1 corresponds to even oscillations at frequency λ for one mode and the length b_2 corresponds to odd oscillations at the same frequency for the same mode, then, the relation $|b_1 - b_2| = \pi(2l - 1)/\mu(\lambda)$ is valid* ($l \in \mathbb{N}$).

Using the first reduction method, we plotted curves of eigenfrequency versus plate length for α-, β, γ-modes with 15, 30, and 15 series terms, respectively (Fig. 2). In Fig. 2, the even modes are denoted by the subscript 0, and the odd modes by 1.

For rather great plate lengths, the eigenvalues calculated by formulas (11) and (12) differ from the eigenvalues of the corresponding modes given in Fig. 2 by less than 0.1. Statements 1 and 2 are proved on the basis of (11) and (12), which are first approximations of the dispersion relation. The data shown in Fig. 2 suggest that Statements 1 and 2 are generally valid.

In the case of α- and γ-modes with a close number of series terms in (9), the eigenvalues for the two reduction methods differ in the 3rd or 4th decimal place. The number of coincident digits increases with increase in the number of series terms. From the numerical experiments it follows that for the α- and γ-modes, the second reduction method converges faster and requires smaller computation time. For the β-modes, both methods converge more slowly and the number of coincident decimal places is smaller than that for the α- and γ-modes. In the first method, to obtain eigenvalues with accuracy up to two decimal places, it suffices to use 21 terms of series (9) for the α- and γ-modes and 28 terms of series (10) for the β-modes.

Figure 3 shows velocity and pressure fields for the α-, β-, and γ-modes in a cross section of the channel for the function $u_1(x, y, z)$ with $x = 2$, $b = 2$, $c = 330$ m/sec, $M = 0$, $H = 1$ m, $t = 0$, and $\lambda = 6.44$, 1.31, and 1.36 for the α, β-, and γ-modes, respectively.

Conclusions. It is shown that three types of natural oscillations near a honeycomb lattice cell exist, which differ from the oscillation modes near one plate in a channel. These oscillations are proved to exist always. Curves of oscillation frequencies versus plate length are constructed. The number of modes is evaluated for each oscillation mode. Approximate relations linking plate lengths corresponding to the same frequency are derived.

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